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On a Local-Search Heuristic for a Class of Tracking Error Minimization Problems in Portfolio Management

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Abstract. In this paper we describe a 2-phase simulated annealing heuristic approach for a special class of portfolio management problems: the problem of optimizing a stock fund with respect to tracking error and transaction costs over time subject to a set of complex constraints with a linear factor return model "feeding" the objective function with data. Our results on managing two real-world funds of a major German capital investment company have shown that this meta-heuristic provides proposals for the fund manager which are feasible with respect to the investment guidelines and excellent in quality in acceptable time. Thus the approach is ideally suited to be used routinely and interactively within a decision support system to assist the fund manager in his complex task of portfolio control and optimization.

Keywords: portfolio optimization, tracking error problem, investment guidelines, multi-factor return model, transaction costs, meta-heuristics, simulated annealing

1. Introduction

In portfolio optimization a portfolio manager is faced with the problem to select from a usually very large set of assets offered in the market (stocks, bonds, options, etc.) a subset of assets for investment following a specific objective with respect to (future) performance and risk while respecting certain constraints from the budget and from legal regulations as well as internal guidelines. Once a portfolio is generated it has to be controlled and re-optimized over time due to the dynamic of asset markets and here another aspect, the minimization or limitation of transaction cost becomes critical. Managing a portfolio can be guided by two strategies: In "active" management we assume that markets are not fully efficient and offering potentials such that a performance exceeding that of standard indices can be achieved by using specific knowledge, i.e. extensive financial analysis and the fund managers experience. On the other hand we have the hypothesis that financial markets are efficient and that return and risk are fully reflected in the actual asset price which leads to the concept of "passive" management and the development of so-called index-tracking models, where portfolio management tries to imitate or copy some benchmark-portfolio, as for instance a market index (DAX, EUROSTOXX, etc.).

Considering the wide range of assets, the heterogeneous group of investors with (individual) different objectives, and the set of investment regulations or guidelines,

* Ulrich Derigs dedicates this paper to Professor Dr. Rainer E. Burkard on the occasion of his 60's birthday.

portfolio management is obviously a non-trivial and non-routine task. In order to reduce complexity and to systematize the problem solution, a multi-stage portfolio management process has been proposed in the literature (see Lederman and Klein (1994), for instance) and for supporting this process a bundle of different models and methods has been developed and there exist several classes of software systems which are used in practice. For *market analysis* highly sophisticated econometric models have been developed and there exist several services which support fund management with up-to-date data on assets. The problem of *asset selection* is the core of portfolio theory, and the famous model of Markowitz (1952) gives a formal theoretical solution. For *controlling investment guidelines* knowledge-based software systems have been developed which check portfolios with respect to the set of constraints and indicate violations. Yet, all these models and support systems leave the (human) portfolio manager with the problem to restructure the portfolio appropriately, i.e. the fund manager has no support in identifying promising transactions which will result in a feasible and profitable portfolio encountering the portfolio's objectives as well as the aspect of minimizing transaction costs. It is this portfolio manager's daily decision problem which has motivated our research and which is the goal of our approach developed for portfolio (re-)optimization.

Portfolio Optimization is a classical research problem in Operations Research. Research started with the mean–variance portfolio selection model by Markowitz (1952) which from a mathematical programming point of view is a quadratic optimization problem with linear constraints. This basic model has several limitations which prohibit its use in practice and in the sequel several extensions and modifications have been developed which address complex constraints, transacting costs or represent alternative management objectives. Besides the problem of representing uncertainty appropriately which is inherent in all investment models all these developments share another obstacle: The more these models are representing relevant practical issues the more they become computationally infeasible. Bienstock (1996) has shown that the classical quadratic optimization problem becomes $\mathcal{NP}$-hard already if adding a single cardinality constraint. Thus, due to the computational complexity of portfolio models the solver has to resume to heuristic, i.e. non-exact methods.

Consequently, in recent years several authors have developed and studied approximative methods especially meta-heuristics in this problem area: Beasley, Meade, and Chang (1999) propose a genetic algorithm for solving a tracking error model with a special constraint for limiting the shares of individual assets. Chang et al. (2000) have analyzed various meta-heuristics for cardinality constrained portfolio optimization problems. Crama and Schyns (1999) have proposed a simulated annealing heuristic for generating mean-variance efficient portfolios subject to a set of complex constraints. In all these approaches a specific problem type is discussed, i.e. the basic model is extended by a specific aspect like for instance the introduction of a cardinality constraint to limit the complexity of the portfolio with respect to the number of assets, the introduction of transaction cost or the definition of another objective like tracking-error and the use of a specific return model. Yet, to our knowledge our work is the first which explicitly focusses on models representing concrete portfolio re-optimization problems occurring in a capital investment firm concerning the objective function as well as investment guidelines.

Our work is motivated by several projects which we have conducted with major German capital investment firms. Not only there is no generally accepted or generally applied model in practice, the models which have been developed in different companies and which are appropriate for the different classes of funds differ significantly with respect to structure, i.e. objective function and constraints such that we cannot expect one algorithm to be suitable for all problems. In Derigs and Nickel (2002) we have developed a concept for a model-based decision support system generator (DSS-generator) for portfolio optimization which is open to be flexibly customized for specific problem instances. The core of the generator is a rule-base for managing the investment guidelines and a (set of) local-search heuristic(s). For representing the diverse classes of guidelines arising in practical problems we have developed and implemented the structural concept of so-called bundle constraints. To be able to treat all kind of constraints and to allow a fast evaluation of moves local search has to be based on a rather simple class of moves. Also, it is essential to define a proper problem-specific classification and procedural handling of constraints, i.e to decide which constraints are relaxed and penalized in the objective, etc.

In this paper we focus on a special class of portfolio optimization problems: the problem of optimizing a stock fund with respect to tracking error and transaction costs over time subject to a set of complex constraints, including a cardinality constraint and static as well as dynamic bundle constraints, with a linear factor return model "feeding" the objective function with data. With this specification the underlying model differs significantly from the models which have been treated in the OR-literature so far: the guidelines extend the list of constraints studied in Chang et al. (2000) and Crama and Schyns (1999) and the tracking measure is significantly different from the one given by Beasley. Moreover we even link the return model to the selection model and thus to the solution procedure. Altogether the complexity of our model as well as its practical relevance goes beyond the models that have been treated in literature.

We describe a 2-phase Simulated Annealing approach and we report extensive numerical experience on two real-world cases. The local search is based on the so-called class of basic moves where the shares of just two assets are adjusted. In the 2-phase heuristic the first phase is to generate a feasible portfolio and the second phase is to find a near-optimal portfolio, and for the two phases we apply a different classification and handling of constraints as well as different objective functions. The guiding strategy of the second phase is to choose two assets at random and then to determine the optimal step size for the associated basic move. Yet, controlled by some parameter, this steepest descent choice is replaced by the non-improving move in the opposite direction. We customize this general approach to the special problem-class in the following aspects: For the objective function, i.e. the tracking error function and the transaction cost function we develop update formulas to allow an efficient evaluation of the basic moves. Also we use the structure of the return model to deduce formulas for the optimal step-size of a basic move.

We show that with this specialization the 2-phase method is adequate and effective to handle the problems from two case studies. Due to several specialities of the application as for instance the influence of the specific return model on the selection model, the approach is not immediately transferable to the models which have been discussed by other authors. Yet, we think that the fundamental strategy of the 2-phase approach should be applicable to other problem classes too, if the heuristic is adequately customized.

The paper is organized as follows: In section 2 we introduce the mathematical model of the problem and in section 3 we specify the heuristic approach which we have developed to solve this model. We have developed and applied this approach in a project to managing two stock funds. Results for these real-world portfolio management problems are presented in section 4.

2. An index tracking model for passive portfolio management

Within passive portfolio management the fund manager wants to select a portfolio, the performance of which is "as close as possible" to a specific benchmark portfolio because the benchmark portfolio itself is violating a set of legal or contractual constraints, permanently or momentarily, and thus is infeasible. In such a situation certain shares have to be replaced by substitutes which have similar performance profile and thus, quite naturally, there arises an optimization problem to approximate the benchmark by a so-called *tracking portfolio*.

A common concept to measure the quality of the approximation is by criteria based on the difference between the (expected) return of the benchmark and the (expected) return of the tracking portfolio. Here the so-called *tracking error* (TE), quantifies the quality of the approximation by a function of the difference between the return of the benchmark and that of the tracking portfolio, and the *tracking error problem* is to minimize this measure (see Rudd (1980), for example). Note, that in the literature there exist alternate index tracking models that combine a tracking error measure with an excess return approach, for example (cf. Beasley, Meade, and Chang (1999)).

In this section we introduce the basic concept of *index tracking*, we formulate the specific quantitative model and we introduce extensions which are necessary for model implementation in real world scenarios. Finally we shortly describe a specific return model which was given in our application study. Under a conceptual viewpoint, the return model is outside the decision model and only feeding the optimization model with data, yet its specific structure determines the instantiation of the objective function of the optimization model and, in our application, it has been motivating the development of our specific heuristic search procedure. Note that several other return models have been assumed in literature, cf. for instance Beasley, Meade, and Chang (1999), which lead to different selection models and these differences with respect to the return model are one obstacle for transferring different methods between these approaches and allowing a comparison on problems given in literature.

The basic model. For representing the problem and stating the model we use the following notation:
N: the number of available assets (i.e. the size of the investment universe),
$S_i(t)$: the price of asset i at time t ($i = 1, \dots, N$),
$y_i \in \mathbb{Z}$: pieces of asset i in a portfolio ($i = 1, \dots, N$),
$S_P(t) := \sum_{i=1}^{N} y_i S_i(t)$: the net asset value (NAV) of a portfolio at time t,
$x_i := y_i S_i(t)/S_P(t)$: the share of asset i in a portfolio ($i = 1, \dots, N$),
r_i: the return of asset i ($i = 1, \dots, N$),
r_x: the return of a portfolio.
Usually *no short sales* of assets are allowed and a *budget constraint* is given. Thus the set X of all potential portfolios represented by shares can be written as

$$X = \left\{ x = (x_1, \dots, x_N) \,\middle|\, \sum_{i=1}^{N} x_i = 1, x_i \geqslant 0, i = 1, \dots, N \right\}. \tag{1}$$

Since the future price $S_i(T)$ of an asset i at time T is usually unknown at present time t, the return r_i as well as the return r_x is also an unknown quantity at time $t < T$. This uncertainty is generally modeled by assuming that r_i is a random variable and the estimation of the probability distribution of r_i is based on so-called *return-models*. The selection of an appropriate return model is a non-trivial problem and is outside the selection model.

Given the *investment universe* $\{1, \dots, N\}$ of N assets let without loss of generality the first N_B assets of the investment universe be the assets contained in the benchmark. Then let $x = (x_1, \dots, x_N)$ represent the tracking portfolio and

$$x^{(B)} = \Big(x_1^{(B)}, \dots, x_{N_B}^{(B)}, \underbrace{x_{N_B+1}^{(B)}, \dots, x_N^{(B)}}_{=0}\Big) \tag{2}$$

represent the benchmark portfolio. The shares $x_{N_B+1}^{(B)}, \dots, x_N^{(B)}$ are set to zero because the corresponding assets are not contained in the benchmark. Now, let r_x and r_B be the (random) returns of x and $x^{(B)}$, respectively, then the *tracking error variance* (TEV), is defined as the variance of the difference $r_x - r_B$, i.e. $TEV(x) := Var(r_x - r_B)$ and we obtain the following optimization problem:

$$\min\big\{TEV(x) \mid x \in X\big\}. \tag{3}$$

This formulation corresponds to the principle of passive fund management. In passive fund management the investor is only exposed to market risk in contrast to active management where he faces individual company risk, too. It is important that also a poor performance of the benchmark is imitated. For example, for an investor who manages his portfolio by a mixed strategy, i.e. integrates active with passive management, it is essential that the passive fund copies the benchmark as precise as possible, since a hedging strategy of the market risk would fail, if the fund manager would try to outperform the benchmark return.

Yet, this basic tracking error model has several limitations which are crucial with respect to practical implementation: The model does not incorporate constraints which exclude portfolios from the set X due to investment guidelines, and it does not incorporate transaction costs into the decision. Modifications of the basic model which address these extensions lead to mathematical programs which are computationally intractable (cf. Bienstock (1996)). In the following we briefly attend to these important issues and we specify our return model.

Constraints. Portfolio selection has to obey investment guidelines as for instance the German law on investment trust companies KAGG (cf. BAKred (1998)). Thus the set X of all potential portfolios stated in the model above has to be restricted to a subset of so-called *feasible portfolios*. From an algorithmic point of view the source of the constraints is irrelevant and only the structure is relevant. The most simple and basic constraints are the so-called *floor/ceiling constraints* for each individual asset ($L_i \leqslant x_i \leqslant U_i$). In order to restrict the shares of a certain group B of assets, as for instance the shares for industrial sectors, a (*static*) *bundle constraint* can be imposed ($L_B \leqslant \sum_{i \in B} x_i \leqslant U_B$). To reduce the complexity of portfolio control and also to control transaction costs, the number of assets in the portfolio is often restricted by a *cardinality constraint* ($N_L \leqslant |\{i \in \{1, \ldots, N\} \mid x_i \neq 0\}| \leqslant N_U$). Yet, there are more complex constraints, as for instance the following rule of the KAGG which requires that "the share of the set of those assets with an individual share of at least 5% should not exceed 40%" ($\sum_{i:x_i \geqslant 0.05} x_i \leqslant 0.4$). Note, that this rule of the KAGG can be formulated as a *(dynamic) bundle constraint* with $B := \{i = 1, \ldots, N \mid x_i \geqslant s_B\}$, $s_B := 0.05$, $L_B := 0$, $U_B := 0.4$. It is dynamic because the selection of the bundle condition can change if the composition of x varies. For a thorough discussion of common types of constraints see Chang et al. (2000) and Crama and Schyns (1999).

Transaction costs. Re-optimizing a portfolio successively over time may result in buy- and sell transactions the costs of which exceed the (expected) gain in performance, i.e. reduce additional return. Consider for simplicity that a constant transaction cost factor c^{var} applies for each amount of ordered assets. Then the total transaction costs to turn over from portfolio $\bar{x}$ to x is $C^{tac} = c^{var} \cdot TO(\bar{x}, x)$ with the turnover volume

$$TO(\bar{x}, x) := \sum_{i=1}^{N} |x_i - \bar{x}_i| \tag{4}$$

and the net return of a portfolio respecting the transaction costs is reduced by an amount which is proportional to the turnover volume. For the straightforward modeling approach to reduce the return by transaction costs the tracking error variance objective is not suitable, since, assuming a constant transaction cost factor, transaction costs have no influence on this measure.

Another standard approach which we follow here is to treat transaction costs as a second objective function (cf. Adcock and Meade (1994)) and to consider portfolios which are efficient with respect to $TEV(x)$ and $TO(\bar{x}, x)$. Such portfolios can be constructed by solving the following parametric optimization problem

$$\min\left\{(1-\lambda)\cdot TEV(x)+\lambda\cdot c_{TO}\cdot TO(\bar{x},x)\mid x\in X\right\} \tag{5}$$

varying the weighting parameter $\lambda \in [0, 1[$. Here $c_{TO} > 0$ is a fixed scalar to balance the magnitude of the two objective functions.

Factor return model. A common approach is to model returns with a multi-factor model (see Sharpe (1970)). Here it is assumed that the returns of assets are dependent on the return of several macro-economical factors which are identified by principal component analysis for example. Applying such a model we assume that the return r_i of an asset i $(i = 1, \ldots, N)$ is (linearly) dependent on the return R_g of K macro-economical factors or indices $(g = 1, \ldots, K)$, i.e.

$$r_i = \alpha_i + \sum_{g=1}^{K} \beta_{ig} R_g + \epsilon_i \tag{6}$$

with:
r_i: return of asset i $(i = 1, \ldots, N)$,
R_g: return of factor g $(g = 1, \ldots, K)$,
α_i: asset-specific component of r_i (independent of factors),
β_{ig}: sensitivity of r_i to the return of factor g (factor loading),
ϵ_i: random perturbation variable.
For given factors the returns R_g are modeled as random variables with expected value $E(R_g)$ and variance $Var(R_g)$ $(g = 1, \ldots, K)$ and the random perturbations are assumed to be standard normally distributed and uncorrelated. Based on these assumptions we obtain

$$Cov(r_i, r_j) = \sum_{g=1}^{K}\sum_{h=1}^{K} \beta_{ig}\beta_{jh} Cov(R_g, R_h) + \underbrace{Cov(\epsilon_i, \epsilon_j)}_{=0 \text{ if } i\neq j}, \tag{7}$$

$$Var(r_i) = \sum_{g=1}^{K}\sum_{h=1}^{K} \beta_{ig}\beta_{ih} Cov(R_g, R_h) + Var(\epsilon_i). \tag{8}$$

The return r_x of a portfolio $x = (x_1, \ldots, x_N)$ is linearly dependent on the K returns R_g $(g = 1, \ldots, K)$, and we obtain

$$\begin{aligned} r_x &= \sum_{i=1}^{N} x_i \left(\alpha_i + \sum_{g=1}^{K} \beta_{ig} R_g + \epsilon_i\right) \\ &= \underbrace{\sum_{i=1}^{N} x_i \alpha_i}_{=:\alpha_x} + \sum_{g=1}^{K} \underbrace{\left(\sum_{i=1}^{N} x_i \beta_{ig}\right)}_{=:\beta_{xg}} R_g + \underbrace{\sum_{i=1}^{N} x_i \epsilon_i}_{=:\epsilon_x}, \end{aligned} \tag{9}$$

$$Var(r_x) = \sum_{g=1}^{K}\sum_{h=1}^{K} \beta_{xg}\beta_{xh}Cov(R_g, R_h) + \sum_{i=1}^{N} x_i^2 Var(\epsilon_i). \tag{10}$$

Using the notation $\beta := (\beta_{ig})_{i=1,\ldots,N;g=1,\ldots,K}$ and $C := (Cov(R_g, R_h))_{g,h=1,\ldots,K}$ for the matrices of factor loadings and covariances, respectively, the tracking error variance $TEV(x)$ is given by:

$$\begin{aligned} TEV(x) &:= Var(r_x - r_B) \\ &= \left(x - x^{(B)}\right)^{\mathrm{T}} \beta C \beta^{\mathrm{T}} \left(x - x^{(B)}\right) + \sum_{i=1}^{N} \left(x_i - x_i^{(B)}\right)^2 Var(\epsilon_i) \end{aligned} \tag{11}$$

and the objective function in (3) is a quadratic function.

Now we are able to formulate the generic class of tracking error minimization problems TEP which we address in this paper:

$$\min \; f(x) := (1-\lambda) \cdot TEV(x) + \lambda \cdot c_{TO} \cdot TO(\bar{x}, x) \tag{12}$$

$$\text{s.t.} \quad L_i \leqslant x_i \leqslant U_i \quad (i = 1, \ldots, N), \tag{13}$$

$$N_L \leqslant \left|\{i \in \{1, \ldots, N\} \mid x_i \neq 0\}\right| \leqslant N_U, \tag{14}$$

$$L_{B_s} \leqslant \sum_{i \in B_s} x_i \leqslant U_{B_s} \quad \text{(for all static bundles } B_s, s = 1, \ldots, N_S), \tag{15}$$

$$\sum_{i: x_i \geqslant s_{B_d}} x_i \leqslant U_d \quad \text{(for all dynamic bundles } B_d, d = 1, \ldots, N_D), \tag{16}$$

$$x \in X. \tag{17}$$

Several modifications of the classical (unconstrained) portfolio selection models which address extensions such as (13)–(15) have been proposed in literature. Yet, these mathematical programs become computational in-tractable soon. Bienstock (1996) has shown that the classical quadratic optimization problem becomes $\mathcal{NP}$-hard already if adding a single cardinality constraint. Due to this computational complexity of our model TEP we have to resume to heuristics, i.e. non-exact methods.

3. A local-search approach for TEP

As we have seen we can formalize the tracking error problem as a general mathematical program

$$\min\left\{f(x) \mid x \in X,\; x \text{ fulfills constraints in } C_0\right\}. \tag{18}$$

Here the objective function is highly non-linear in general and the constraints may define a non-convex, possibly even discrete set of feasible solutions. For such problems exact and analytical methods are infeasible and only heuristic methods are applicable. In recent years the concept of *meta-heuristics* has been developed to cope with such highly complex optimization problems. A meta-heuristic is an iterative master process that guides and modifies the operations of “subordinate heuristics” to efficiently produce high-quality solutions (cf. Osman (1995)). Prominent examples for meta-heuristics are *Simulated Annealing (SA)*, *Tabu Search (TS)* etc. (cf. Aarts and Lenstra (1997)). For the subordinate heuristic the concept of *neighborhood search* or *local search* is most common.

In this section we describe a 2-phase SA-based heuristic for solving TEP. We will first specify the overall design and then describe the implementation of the two phases separately.

3.1. *Design specification*

For the computational efficiency of a specific local search implementation as well as for its effectiveness and quality three issues are crucial: the handling of the constraints from the underlying optimization model, the definition of a neighborhood structure allowing an efficient evaluation of moves and the choice of the heuristic search strategy. We start the description of our approach focusing on these issues.

Handling of constraints. When solving a (portfolio) optimization problem by local search the set C_0 of constraints is partitioned into three classes, i.e. $C_0 = C_1 \cup C_2 \cup C_3$ with the following property:

– The constraints in set C_1 are controlled through a so-called *constraint checker*. The constraint checker is basically a function $Check(x, C_k)$ which expects as input a portfolio x and returns the value `TRUE` if all constraints in C_k are fulfilled by x, and the value `FALSE` otherwise.

– The constraints in C_2 are relaxed, i.e. violations are evaluated and introduced as penalty term into the objective function of the model leading to the modified objective function, the so-called *evaluation function*:

$$F(x) := f(x) + Penalty(x, C_2). \tag{19}$$

This penalty approach is a common trick in constrained optimization. Here the fulfillment of the relaxed constraints has to be realized by the optimization process, i.e. due to the relative high costs of infeasible solutions, the solution process should finally converge to feasible solutions only.

• Finally, the constraints in C_3 are controlled through the specific local search procedure, i.e. only portfolios which are feasible with respect to C_3 are generated during the search.

Note, that when optimizing a specific portfolio, this partition has to be appropriately specified and the necessary *Check*-functions and evaluation functions have to be activated. Note, that although a *Check*-function may be available which is able to control all constraints in C_0, as for instance a professional guideline management system, it may be more efficient to work with non-empty sets C_2 and C_3. In our 2-phase approach we will even change the partition in the course of the solution process.

Neighborhood structure. A crucial choice within every local search approach is the appropriate specification of a neighborhood structure. Here it is essential that there exists a fast algorithm to calculate the effect of a move from a solution x to a solution x' from the neighborhood $N(x)$. See for instance the application of local search to the quadratic assignment problem (cf. Burkard and Rendl (1984)). Due to the complexity of the *TEV*-function (11) a straightforward evaluation of $f(x')$ and $F(x')$, respectively, is computationally infeasible since it has a complexity of $\mathcal{O}(N^2)$. Thus through the definition of an appropriate neighborhood topology the moves have to be restricted allowing a more efficient update formula.

In our approach we restrict the neighborhood to those portfolios which differ in the share of two assets only, i.e. we allow only move-operators $x \to x'$ with $x' \in N(x)$ which increment the share of asset i by δ and reduce the share of asset j by δ, i.e.

$$x' = x + \delta \cdot e_{ij} = x + \delta \cdot (0, \ldots, 0, +1, 0, \ldots, 0, -1, 0, \ldots, 0)^{\mathrm{T}}. \tag{20}$$

Thus, such a *basic move* can be identified by the tripel (δ, i, j). Now, let $D := \beta C \beta^{\mathrm{T}}$ and $V_k := Var(\epsilon_k)$ for $k = 1, \ldots, N$, then it can be shown that the objective values of a neighbor x' can be obtained from the objective values of x by the formulas

$$TEV(x') = TEV(x + \delta e_{ij}) = TEV(x) + UpdateTEV(\delta), \tag{21}$$
$$TO(\bar{x}, x') = TO(\bar{x}, x + \delta e_{ij}) = TO(\bar{x}, x) + UpdateTO(\delta) \tag{22}$$

with

$$\begin{aligned} UpdateTEV(\delta) = {} & 2\delta \sum_{k=1}^{N} \big(x_k - x_k^{(B)}\big)(D_{ki} - D_{kj}) + \delta^2(D_{ii} - 2D_{ij} + D_{jj}) \\ & + \delta\big(2x_i - 2x_i^{(B)} + \delta\big) \cdot V_i - \delta\big(2x_j - 2x_j^{(B)} - \delta\big) \cdot V_j, \end{aligned} \tag{23}$$
$$UpdateTO(\delta) = |x_i + \delta - \bar{x}_i| - |x_i - \bar{x}_i| + |x_j - \delta - \bar{x}_j| - |x_j - \bar{x}_j|. \tag{24}$$

Here, *UpdateTEV* is a quadratic function which is computable in order $\mathcal{O}(N)$. A simple case differentiation gives:

$$UpdateTO(\delta) = \begin{cases} -2(\delta - \delta_1); & \delta < \delta_1, \\ 0; & \delta_1 \leqslant \delta \leqslant \delta_2, \\ 2(\delta - \delta_2); & \delta_2 < \delta \end{cases} \tag{25}$$

with

$$\delta_1 := \begin{cases} x_j - \bar{x}_j; & x_i \leqslant \bar{x}_i \wedge x_j \leqslant \bar{x}_j, \\ \min\{\bar{x}_i - x_i, x_j - \bar{x}_j\}; & (x_i \leqslant \bar{x}_i \wedge x_j > \bar{x}_j) \vee (x_i > \bar{x}_i \wedge x_j \leqslant \bar{x}_j), \\ \bar{x}_i - x_i; & x_i > \bar{x}_i \wedge x_j > \bar{x}_j \end{cases} \tag{26}$$

and

$$\delta_2 := \begin{cases} \bar{x}_i - x_i; & x_i \leqslant \bar{x}_i \wedge x_j \leqslant \bar{x}_j, \\ \max\{\bar{x}_i - x_i, x_j - \bar{x}_j\}; & (x_i \leqslant \bar{x}_i \wedge x_j > \bar{x}_j) \vee (x_i > \bar{x}_i \wedge x_j \leqslant \bar{x}_j), \\ x_j - \bar{x}_j; & x_i > \bar{x}_i \wedge x_j > \bar{x}_j. \end{cases} \quad (27)$$

Thus *UpdateTO* is a "funnel-shaped" function which can be computed in $\mathcal{O}(1)$ and the update for the general objective function (12)

$$Update(\delta) := (1 - \lambda) \cdot UpdateTEV(\delta) + \lambda \cdot c_{TO} \cdot UpdateTO(\delta) \quad (28)$$

can be computed in linear time.

The meta-heuristic search strategy. For guiding the local search procedure we have chosen the strategy of *Simulated Annealing* (SA). SA is a local search concept which performs a stochastic neighborhood search of the solution space. The advantage of SA over classical local search methods like the steepest descent method is its ability to avoid getting trapped into bad local minima. The motivation for using SA here is its rather simple implementation and the good experience with SA for solving other quadratic optimization problems.

The underlying principle arises from an analogy with certain thermo-dynamical processes, i.e. the cooling of melted solid (cf. Kirkpatrick, Gelatt, and Vecchi (1983)) and is rather simple: Starting from an initial solution x, another solution $x' \in N(x)$ is selected randomly. If x' improves the value of the objective function, i.e. $F(x') < F(x)$, then x' is accepted, i.e. x' replaces x as the new current solution. Otherwise, x' is accepted with a certain probability only. Then the next iteration is started from the current solution x.

During the process the probability of accepting a non-improving solution decreases with the magnitude of the deterioration $\Delta = f(x') - f(x)$ and with the duration of the process, i.e. the number of iterations which have been performed. The process of controlling this probability is called the *cooling schedule* and is generally governed by the Boltzmann-function $p(T, \Delta) = \exp(-\Delta/T)$, i.e. the neighbor x' is accepted with probability $p(T, \Delta)$. Here T is called the *temperature* which is a non-decreasing function in the number of iterations. In the so-called *geometric cooling schedule* the temperature T is kept constant during a fixed number L of consecutive selections/moves, a so-called (cooling) *stage* of the process. After each stage the temperature T is multiplied by a constant factor $\alpha \in]0, 1[$. In our tests we have implemented a geometric cooling schedule with $\alpha = 0.95$.

For the specification of the initial temperature we have implemented the following data-dependent procedure (cf. Johnson et al. (1989)): During a first phase SA is run for a pre-specified number L' of iterations without rejecting any move and the average deterioration Δ of the objective function over this first phase is calculated. We then calculate the temperature T_0 which would have given a pre-specified acceptance probability χ_0 for the moves taken during the first phase, i.e. $T_0 = -\Delta/\ln \chi_0$. We have implemented this first phase approach with $L' = 400$ iterations (random moves) and $\chi_0 = 0.8$. Also, we have implemented the common criteria to stop the SA-procedure after a pre-specified number of K stages.

Usually, SA is very sensitive to the choice of the starting solution. Therefore, in our implementation, we allow to handle a population of (initial) portfolios and then we perform the SA heuristic for each of these solutions independently (i.e. without any interaction). After termination we keep the best solution of the entire set of solutions which have been constructed during the search processes.

3.2. The 2-phase approach for solving TEP

Heuristic approaches for solving hard optimization problems usually consist of two phases: in phase I (construction phase) a feasible solution is generated and this solution is passed to the second phase to be used as initial solution and in phase II (improvement phase) better solutions are generated while feasibility is maintained using some local search strategy. The basic idea of our approach for solving TEP is to apply Simulated Annealing in both phases. Yet, for solving the feasibility problem in phase I we use a different objective function and neighborhood definition as for the optimization problem in phase II. In the following we describe the specification of both phases, i.e. the partitioning of C_0 and the specification of the basic moves, as well as the modified objective function for phase I.

3.2.1. Description of phase I

Partitioning of constraints and definition of penalty function. For phase I we partition the set C_0 as follows: C_1 consists of the floor/ceiling constraints and C_3 contains the cardinality constraint and the budget constraint. C_2 consists of all other (dynamic and static) bundle constraints. In order to guide the heuristic search procedure to construct solutions fulfilling the bundle constraints in C_2 we penalize all violations and obtain the following evaluation function

$$F(x) := f(x) + \sum_{c \in C_2} \lambda_c \cdot \max\left\{0, \sum_{i \in B_c} x_i - U_{B_c}, L_{B_c} - \sum_{i \in B_c} x_i\right\} \tag{29}$$

with B_c the set of assets defining the bundle of constraints $c \in C_2$. In the computational results reported in this paper we have penalized all violations of guidelines $c \in C_2$ with the same scalar $\lambda_c := 40$.

A necessary and first step in most local search applications is the generation of a feasible start solution. Thus, for applying SA in the construction phase we have to generate an initial portfolio which fulfills all the constraints in $C_1 \cup C_3$. For that purpose we have implemented the following procedure.

Construction of a start solution for phase I. First we determine the set $I_R := \{1, \ldots, N \mid L_i > 0\}$ of required assets. If $\sum_{i \in I} L_i > 1$ or $|I_R| > N_U$ then the problem is infeasible. Otherwise we determine a set $K \subset \{1, \ldots, N\} \setminus I_R$ of minimal cardinality such that $\sum_{i \in I_R \cup K} U_i \geqslant 1$ by successively selecting those indices i from $\{1, \ldots, N\} \setminus I_R$ with the largest U_i. If $|I_R| + |K| > N_U$ then the problem is infeasible.

Otherwise we set $N_{\min} := \max\{|I_R| + |K|, N_L\}$, select randomly a number $j \in \{N_{\min}, \ldots, N_U\}$ and extend the set I_R by a set J of $j - |I_R|$ different assets chosen from the set $\{1, \ldots, N\} \setminus I_R$. Let $I := I_R \cup J$. If

$$\sum_{i \in I} U_i \geqslant 1 \tag{30}$$

then the procedure terminates successfully, otherwise we modify I replacing assets from the set $J \setminus K$ by assets from the set $K \setminus J$ until this condition (30) is fulfilled.

If $\sum_{i \in I} L_i = \sum_{i \in I} U_i$ we set $x_i = L_i$ for all $i \in I$. Note that for this case we have $\sum_{i \in I} L_i = 1 = \sum_{i \in I} U_i$ and thus we have no optimization problem at all.

Otherwise all assets $i \in I$ are set to their lower bound L_i and we set $L^{sel} := \sum_{i \in I} L_i$ and the "slack" $S^{sel} := \sum_{i \in I} (U_i - L_i)$. Now, in order to fulfill the budget constraint the remaining share $1 - L^{sel}$ is allocated to the assets $i \in I$ by the following formula:

$$x_i = L_i + (U_i - L_i) \cdot \frac{1 - L^{sel}}{S^{sel}}. \tag{31}$$

This simple construction procedure guarantees the fulfillment of the floor/ceiling constraints in C_1 and the cardinality constraint and budget constraint in C_3.

Specification of the basic move. In a first step two different assets i, j are selected randomly. If both assets are not contained in the current portfolio, i.e. $x_i = 0$ and $x_j = 0$, then we decide not to perform a basic move (20) and we have to try another pair of assets. Otherwise we have to distinguish between three cases. In the case that both assets are contained in the current portfolio (stated as *Case 1*), we select at random a new feasible share for asset i and modify the share of asset j to compensate this change. For each of the possibilities that exactly one asset, i or j, has a share greater than zero (stated as *Case 2* and *Case 3*, respectively), two sub-cases have to be distinguished. If the current portfolio has a cardinality less than the upper bound N_U, we apply the procedure of *Case 1*. Otherwise we simply exchange the shares of the two assets. Finally, the potential move is checked with respect to the constraints in C_1. In table 1 a pseudo-code for this move-operator is given.

3.2.2. Description of phase II

Partitioning of constraints. Now, for the optimization in phase II the set C_0 is partitioned such that C_2 is empty, C_1 contains the cardinality constraint and C_3 consists of all other constraints, i.e. the budget constraint, the floor/ceiling constraints and the (dynamic and static) bundle constraints. Thus the move-operator of phase II has to maintain feasibility with respect to all these constraints and the evaluation function is the actual objective function, i.e. $F(x) = f(x)$.

Table 1
Procedure for the *Phase1-Move*.

```
Phase1_Move(x, x'):
repeat
    x' := x
    select i, j ∈ {1, ..., N} randomly with i ≠ j
    if (x_i > 0 or x_j > 0) then
        if (x_i > 0 and x_j > 0) then                          (Case 1)
            select x'_i ∈ [L_i, U_i] randomly
            x'_j := x_j + (x_i − x'_i)
        if (x_i = 0 and x_j > 0) then                          (Case 2)
            Swap(i, j)
        if (x_i > 0 and x_j = 0) then                          (Case 3)
            if |{x_k ≠ 0 | k = 1, ..., N}| < N_U then          (Sub-case 1)
                select x'_i ∈ [L_i, U_i] randomly
                x'_j := x_j + (x_i − x'_i)
            else                                               (Sub-case 2)
                x'_i := 0
                x'_j := x_i
until (Check(x', C_1) = TRUE and x' ≠ x)
```

Specification of the basic move. The general strategy in phase II is to choose the assets for a basic move at random and then to determine the "optimal" step size δ, i.e. to minimize the corresponding Update-function with respect to the constraints. Yet, this "greedy" principle is combined with some random argument allowing non-improving moves. For that purpose we introduce a control parameter $p \in [0, 1]$. A pseudo-code description of the move-operator is given in table 2.

In a first step a pair of different assets i, j with at least one already contained in the current portfolio, i.e. with $x_i > 0$ or $x_j > 0$, is selected randomly. Then procedure "Calculate_Max_Delta" determines values δ^- and δ^+ for the maximal possible decrease and increase of the current share x_i such that feasibility with respect to the constraints in C_3 is maintained by the basic move and procedure "Calculate_Opt_Delta(δ^*)" determines the step size δ^* giving the minimum of $Update(\delta)$, i.e. the "optimal" step size.

In the case that δ^* is in $[-\delta^-, \delta^+]$ with $\delta^* \neq 0$ this "most improving" step size is applied. If $\delta^* \geqslant \delta^+$ and $\delta^+ = 0$ then we set $\delta^* := -\delta^-$, analogously, if $\delta^* \leqslant -\delta^-$ and $\delta^- = 0$ then we set $\delta^* := \delta^+$. Thus, in these cases where we are not allowed to move into the favorable direction we force to take a "large" step in the opposite direction to escape from this "deadlock" situation and we generate a neighboring solution for the SA-metaheuristic selection. Otherwise, if δ^* exceeds δ^+ and $\delta^+ > 0$, we set $\delta^* := \delta^+$ with probability p and set δ^* to the opposite extremal value $-\delta^-$ with probability $(1-p)$. Analogously, if $\delta^* < -\delta^-$ and $\delta^- > 0$ then we set $\delta^* := -\delta^-$ with probability p and set δ^* to the opposite extremal value δ^+ with probability $(1 - p)$. With this strategy not always to perform the largest feasible improving step we allow to generate non-improving moves for the SA-heuristic.

Table 2
Procedure for the *Phase2-Move*.

```
Phase2_Move(x, x', p):
    x' := x
    repeat
        δ* := 0
        select i, j ∈ {1, ..., N} randomly with i ≠ j
        if (x_i > 0 or x_j > 0) then
            Calculate_Max_Delta(δ⁻, δ⁺)
            Calculate_Opt_Delta(δ*)
            if (δ* = 0) then
                if (δ⁺ = 0) then δ* := −δ⁻ else δ* := δ⁺
            if not (δ* ∈ [−δ⁻, δ⁺]) then
                draw a uniform distributed random number u from [0, 1[
                if (δ* > δ⁺) then
                    if (u > p or δ⁺ = 0) then δ* := −δ⁻ else δ* := δ⁺
                else
                    if (u > p or δ⁻ = 0) then δ* := δ⁺ else δ* := −δ⁻
    until δ* ≠ 0
    x'_i := x_i + δ*
    x'_j := x_j − δ*
```

Now we describe the implementation of the two procedures "Calculate_Max_Delta" and "Calculate_Opt_Delta" successively in this order.

Given a pair of assets i, j the determination of δ^- and δ^+ is performed in two steps. First, for all N_S static bundle constraints we define indicator variables $h_{l,b}$ with $h_{l,b} = 1$ if asset l is contained in bundle constraint b, otherwise $h_{l,b} = 0$ ($l = 1, \ldots, N$; $b = 1, \ldots, N_S$). Note that for the case that both of the assets i, j of a basic move are in a bundle the fulfillment of the bundle condition need not be checked since the increase of the bundle weight by the increase of one asset is compensated by the corresponding decrease of the other asset et vice versa. Now we determine:

$$\begin{aligned}
\delta_i^- &:= \min\left\{(x_i - L_i), \min\left\{\sum_{l=1}^{N} h_{l,b}x_l - L_b \,\middle|\, h_{i,b} = 1 \wedge h_{j,b} = 0,\ b = 1, \ldots, N_S\right\}\right\},\\
\delta_i^+ &:= \min\left\{(U_i - x_i), \min\left\{U_b - \sum_{l=1}^{N} h_{l,b}x_l \,\middle|\, h_{i,b} = 1 \wedge h_{j,b} = 0,\ b = 1, \ldots, N_S\right\}\right\},\\
\delta_j^- &:= \min\left\{(x_j - L_j), \min\left\{\sum_{l=1}^{N} h_{l,b}x_l - L_b \,\middle|\, h_{i,b} = 0 \wedge h_{j,b} = 1,\ b = 1, \ldots, N_S\right\}\right\},\\
\delta_j^+ &:= \min\left\{(U_j - x_j), \min\left\{U_b - \sum_{l=1}^{N} h_{l,b}x_l \,\middle|\, h_{i,b} = 0 \wedge h_{j,b} = 1,\ b = 1, \ldots, N_S\right\}\right\}
\end{aligned} \tag{32}$$

and we define $\delta^- := \min\{\delta_j^+, \delta_i^-\}$ and $\delta^+ := \min\{\delta_i^+, \delta_j^-\}$.

In a second step, these two values may have to be reduced to prevent violations of the N_D dynamic bundle constraints.

For this purpose we analyze the N_D dynamic bundle constraints one after the other and update (more precisely: reduce) δ^- and δ^+ if necessary. Let $b \in \{B_d \mid d = 1, \ldots, N_D\}$ a dynamic bundle with $A_b := \sum_{k:x_k \geqslant s_b} x_k$ its current weight and $A_b(\delta)$ its weight after performing a basic move with step size δ, also let $\epsilon > 0$ a small number. Note that in a formal analysis we have to discuss changes of $\delta > 0$ only, i.e. we may assume that x_i is increased and x_j is decreased by δ, since for the other case we may swap the index.

Thus we have to distinguish four (basic) cases which lead to several sub-cases for which a different update-formula occurs. The graphs of $A_b(\delta)$ for these cases are displayed in figure 1 and the associated update formulas for δ^+ are given. In our implementation we have set $\epsilon := 10^{-7}$.

The procedure "Calculate_Opt_Delta" determines δ^* for which *Update*(δ) attains its minimum. Since *UpdateTO* is not differentiable at δ_1 and at δ_2 also *Update* is not differentiable there. Yet, *UpdateTO* is linear over each of the three intervals defined by δ_1 and δ_2 and thus we can determine the minima of *Update* on each of the three intervals by applying simple calculus and then identify δ^*. It can be shown that

$$\begin{aligned}\delta^* &= \frac{-(1-\lambda)\cdot\left[\sum_{k=1}^{N}(x_k - x_k^{(B)})(D_{ki} - D_{kj}) + (x_i - x_i^{(B)})V_i - (x_j - x_j^{(B)})V_j\right]}{(1-\lambda)(D_{ii} - 2D_{ij} + D_{jj} + V_i + V_j)} \\ &\quad - \frac{\lambda\cdot(c_{TO}\cdot q)}{(1-\lambda)(D_{ii} - 2D_{ij} + D_{jj} + V_i + V_j)} \\ &= -\frac{\sum_{k=1}^{N}(x_k - x_k^{(B)})(D_{ki} - D_{kj}) + (x_i - x_i^{(B)})V_i - (x_j - x_j^{(B)})V_j}{D_{ii} - 2D_{ij} + D_{jj} + V_i + V_j} \\ &\quad - \frac{\lambda}{1-\lambda}\cdot\frac{c_{TO}\cdot q}{D_{ii} - 2D_{ij} + D_{jj} + V_i + V_j} \end{aligned} \tag{33}$$

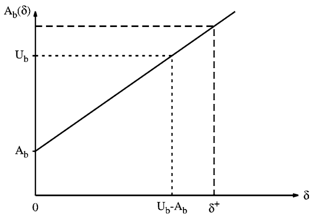
with

$$q := \begin{cases} -1; & \delta < \delta_1, \\ 0; & \delta_1 \leqslant \delta \leqslant \delta_2, \\ 1; & \delta_2 < \delta. \end{cases} \tag{34}$$

In figure 2 we have plotted sample graphs of the three update functions with the points δ_1, δ_2 and the resulting optimal δ^*.

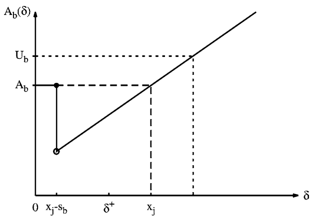
Remarks. In our computational study we have implemented and tested several modifications of the 2-phase approach. In a first approach we applied the basic move from phase II, i.e. choosing an exchange and minimizing *Update*, from the beginning with the penalized objective function and we would then switch to the original objective function as soon as a feasible solution was generated. This approach resulted in a faster minimization of the tracking error variance but it failed to reduce the penalty term *Penalty*(x, C_2)

(Case 1) $x_i \geqslant s_b \wedge x_j < s_b$:



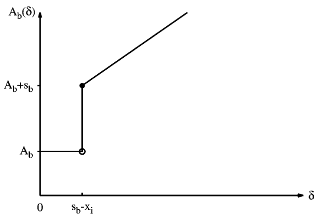
Thus we obtain the simple update formula:
$\delta^+ = \min\{\delta^+, U_b - A_b\}$.

(Case 2) $x_i \geqslant s_b \wedge x_j \geqslant s_b$:



Since x'_j is not allowed to become negative, we have $\delta^+ \leqslant x_j$ and $A_b(\delta^+) \leqslant A_b$ and thus for this case the bundle has no influence on the size of δ^+.

(Case 3) $x_i < s_b \wedge x_j < s_b$:

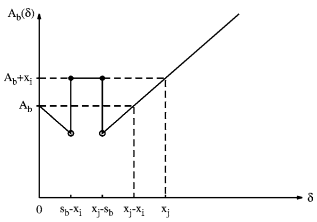


Depending on the upper bound U_b we have to distinguish two cases which lead to the following update formula:

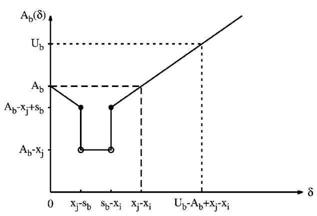
$$\delta^+ := \begin{cases} \min\{\delta^+, U_b - A_b - x_i\}; & U_b \geqslant A_b + s_b, \\ \min\{\delta^+, s_b - x_i - \epsilon\}; & U_b < A_b + s_b. \end{cases}$$

(Case 4) $x_i < s_b \wedge x_j \geqslant s_b$:

As depicted in the following two plots we have to distinguish two sub-cases depending on whether x_i reaches the bound s_b before x_j does (4a), or vice versa (4b):



(4a) $x_j - s_b \geqslant s_b - x_i$:
If $U_b < A_b + x_i$ then we have to apply the update formula $\delta^+ = \min\{\delta^+, s_b - x_i - \epsilon\}$.
If $U_b \geqslant A_b + x_i$ then with the same argument as in Case 2 the bundle has no influence on the size of δ^+.



(4b) $x_j - s_b < s_b - x_i$:
Here we obtain the simple update formula:
$\delta^+ = \min\{\delta^+, U_b - A_b + x_j - x_i\}$.

Figure 1. Four cases with the resulting update formulas for δ^+.

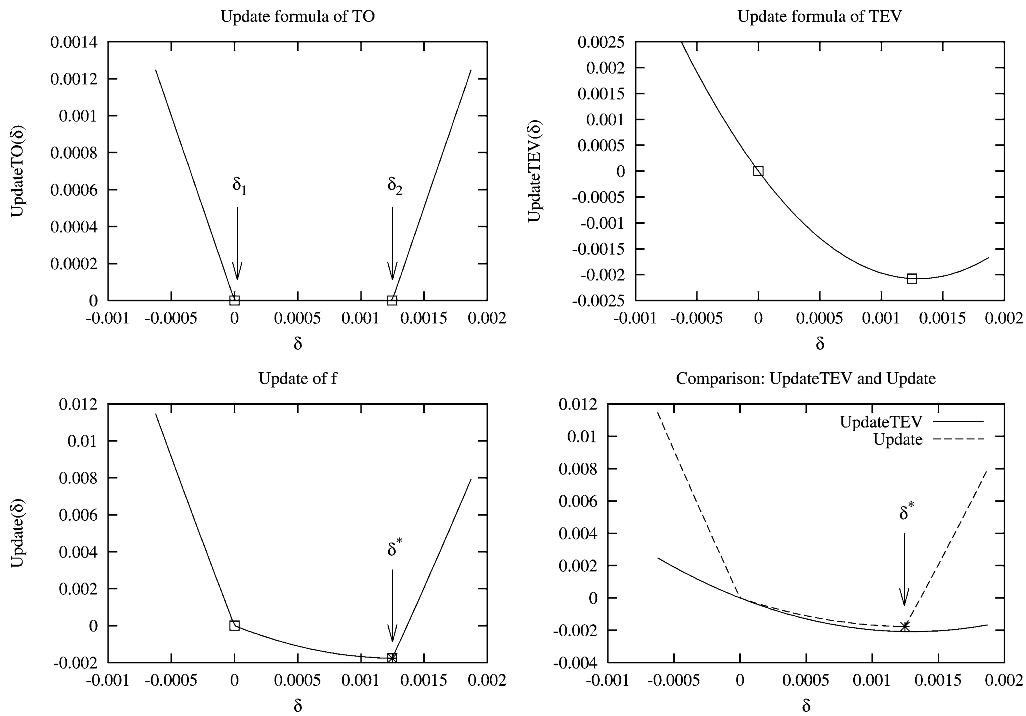


Figure 2. Components of the general update function (28).

to zero and thus this procedure did not converge to a feasible solution. Actually, this experience has been the motivation for developing the 2-phase approach.

4. Application to real-world index tracking problems

In this section we will describe the application of our meta-heuristic approach to two portfolio management problems of a major German investment trust company, for confidentiality called ABC in the sequel. In the first case we report detailed results with ABC-German, an equity fund offered by ABC which we have analyzed and optimized on a daily basis over a period of nearly 3 months. The second case is constructed from a global and multi-currency equity fund ABC-Global of ABC which we have optimized analogously to ABC-German. We do not present the actual results of this optimization here but use the scenario to evaluate the quality of our heuristic approach: Due to the large number of assets in the investment universe the only dynamic bundle constraint contained in the investment guidelines, i.e. the KAGG-40%-rule, is generally uncritical and therefore neglecting the cardinality constraint too, a relaxed TEP can be solved as a *Quadratic Program* (QP) by standard mathematical programming software. Thus we compare the solution of our heuristic with the optimal QP-solution on 43 instances constructed from ABC-Global which have been modified artificially imposing a large number of additional complicating static bundle constraints.

For managing both funds we were given a cost factor of 20 bps for the variable transaction costs, i.e. $c^{var} := 0.002$, and we have set the scalar $c_{TO} := 50$ to balance the magnitude of the two components in the objective function: *TEV* and *TO*.

For the two case problems our 2-phase SA-approach has shown to be rather robust with respect to the choice of a starting solution and thus we did not have to generate a large population. The computational results reported in this paper were obtained specifying the size of the population to 2, the number of iterations per stage $L := 1500$ and the stop parameter $K := 400$ for phase I and for phase II we have specified the size of the population to 1, the number of iterations per stage $L := 2000$ and the stop parameter $K := 400$. Finally we have set the probability parameter of the move-operator $p := 0.75$.

Note that we have performed numerical tests with different parameter settings. We did not intend to determine the "best" configuration. After all, the values chosen seem to be an appropriate and robust choice for our application. When applying the 2-phase approach to other problem-classes, the quality of the choice should be verified empirically. All computations were performed on a Pentium III, 1000 MHz with 256 MB RAM.

4.1. Case I: ABC-German

ABC offers the passively managed equity fund ABC-German which should reproduce the German Stock Index DAX30. Several constraints of the KAGG and some ABC-German-specific guidelines prohibit that the benchmark DAX30 can be identically reproduced and the portfolio manager has the possibility to introduce a limited number of non-DAX30 stocks contained in the STOXX200-Index into the portfolio. Additionally, a cash position and a position in a future on the DAX30 is possible in order to invest temporary cash flows to the fund. These two non-stock assets, the DAX30 stocks and the STOXX200 stocks, form the investment universe for ABC-German. The standard unit for monetary terms is Euro. Note, that for reasons of confidentiality we have scaled the NAV of ABC-German to 25,000,000 Euro for May 1st, 2001.

For the estimation of the returns the professional QUANTEC *Cross-Country Model*[1] is used, a factor model based on 43 macro-economical factors. On the first trading day of a month QUANTEC provides the factor loadings and the standard deviations of the residual return for each asset of the investment universe as well as the matrix of covariances between the factors. These parameters are valid for the whole month. Due to missing factor loadings in the QUANTEC model eleven assets from the STOXX200 have to be eliminated from the ABC-German universe. Also, most of the $N_B = 30$ stocks of the DAX30-benchmark are contained in the STOXX200 too, so the investment universe for ABC-German consists of $N = 202$ assets.

In table 3 we introduce the set of relevant KAGG-constraints and ABC-German-specific constraints (IG_1)–(IG_9). With respect to the KAGG-constraints this is a rather simple application since the management of ABC-German does not allow complex derivatives, etc.

Table 3
Relevant KAGG-constraints and fund-specific constraints for ABC-German.

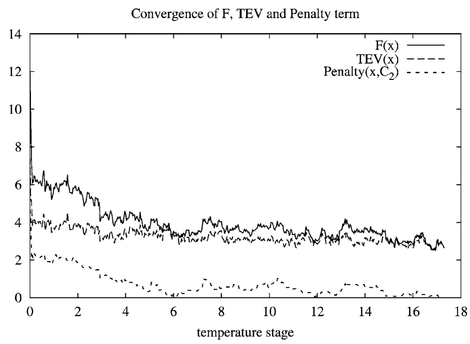
Floor/ceiling constraints:	
(IG_1)	No short-sales, i.e. $x_i \geqslant 0$ ($i = 1, \ldots, N$).
(IG_2)	The share of DAX30-futures should not exceed 20%, i.e. $x_{Fut} \leqslant 0.2$.
(IG_3)	No stock is allowed to have a share of more than 10%, i.e. $x_i \leqslant 10\%$ ($i = 1, \ldots, N-2$).
(IG_4)	The tracking portfolio x should be linked to the benchmark $x^{(B)}$ such that for each DAX30 asset i the share in the portfolio should differ from the share in the actual DAX30 and from the upper bound of 10%, respectively, by at most $\Delta = 0.015$. Since the actual DAX30 shares $x_i^{(B)}$ and Δ are constant parameters these guidelines can be represented as a set of simple floor/ceiling constraints.
(IG_5)	In order to have enough liquidity if fund shares are returned, the cash position should range between 3% and 10%, i.e. $0.03 \leqslant x_{Cash} \leqslant 0.1$.
Dynamic bundle constraints:	
(IG_6)	The share of the set of those assets with an individual share of at least 5% should not exceed 40%, i.e. $\sum_{i:x_i \geqslant 0.05} x_i \leqslant 0.4$.
Static bundle constraints:	
(IG_7)	The share of assets of the automobile industry (BMW, DaimlerChrysler, MAN, Volkswagen) are bounded to vary between 15% and 20% giving another bundle condition.
(IG_8)	Finally, Siemens and Infineon are interpreted as one asset, i.e. the sum of the associated shares should not exceed 0.1.
Cardinality constraint:	
(IG_9)	In order to control management/transaction costs we introduce a cardinality constraint with $N_L := 50$ and $N_U := 150$.

In the following we first present results for the re-optimization for one specific day in order to illustrate the characteristics of our heuristic approach. Then we show the result of managing the fund over a period of about 3 months, from May 1st until July 18th, 2001. Here, we first assume that transaction costs are irrelevant. Finally, we show how the consideration of transaction costs influences the solution. Note that we can give here only the essence of the results on an aggregated level. More detailed information can be found in tables 4–6.

4.1.1. Optimization for a single day

In this subsection we show the results for solving TEP for June 1st, 2001, without consideration of transaction costs, i.e. setting $\lambda := 0$. On May 31st a feasible portfolio was constructed with a *TEV*-value of 0.781, which is a reasonable good quality. Yet, changes in the DAX30 benchmark result in violations of IG_6 and IG_7 making this portfolio infeasible for June 1st. Starting from the population of the initial (two) solutions phase I needs about 3 CPU-seconds for $18 \cdot 1500 \cdot 2 = 54{,}000$ moves for generating a feasible portfolio. Phase II needs about 53 seconds for the $300 \cdot 2000 \cdot 1 = 600{,}000$ moves while generating 3,193 improving feasible solutions within the search process. The best portfolio has a cardinality of 78 assets and an (objective function) value $TEV = 0.817$. The turnover volume is $TO = 0.0573$, i.e. according to the definition (4) of TO nearly 3 percent of the portfolio is restructured.

Convergence of the SA in phase I:



Convergence of the SA in phase II:

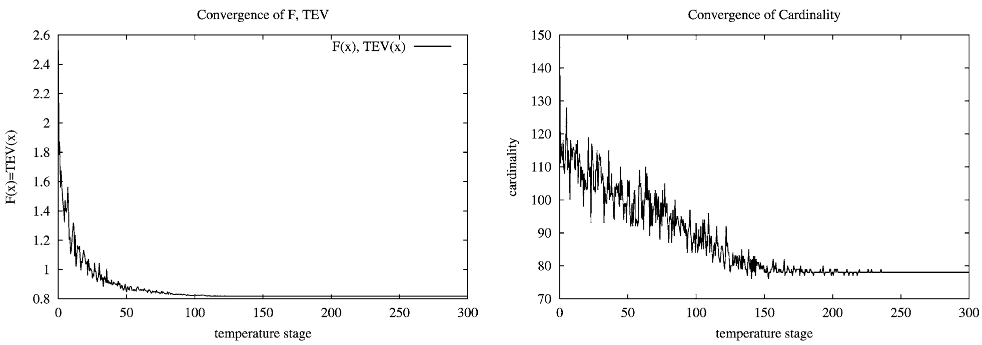


Figure 3. Convergence of the 2-phase approach for June 1st, 2001 (with $\lambda = 0$).

In figure 3 we display the convergence of the evaluation function F and both components of the objective function f, i.e. *TEV* and *TO*. Additionally we show the development of the penalty term and the cardinality for phase I and phase II, respectively. This behaviour has shown to be typical for all SA-runs of the two phases.

The plots demonstrate quite clearly that the SA-search strategy in phase I manages to drive the penalty term to zero, i.e. converges to a feasible portfolio. In phase I the cardinality of the portfolio increases to N_U (= 150) rapidly and maintains at this level. Then in phase II, as can be seen from the plots, the tracking error variance as well as the cardinality are reduced.

Considering transaction costs, i.e. solving the model with $\lambda := 0.25$, the characteristics of the resulting diagrams are similar. Here the tracking error variance of the best solution at the end of phase II amounts to $TEV = 0.850$ but the turnover volume is significantly reduced from 0.0573 to 0.00199 (which equals about 97 Euro).

4.1.2. Managing ABC-German over time

In the following we summarize the results for managing ABC-German over the period from May 1st, 2001 up to July 18th, 2001, i.e. a period of 56 consecutive trading days. Detailed results are listed in table 4.

Table 4
Results for ABC-German (heuristic with λ = 0 and λ = 0.25).

Day	λ = 0							λ = 0.25						
	Current portfolio			"Optimized" portfolio				Current portfolio			"Optimized" portfolio			
	TEV	NAV	fea	*TEV*	NAV	#	C^{tac}	*TEV*	NAV	fea	*TEV*	NAV	#	C^{tac}
05.01				0.623	24,999	57					0.623	25,004	57	
05.02	0.721	24,766	no	0.828	24,767	82	6,469	0.721	24,771	no	0.662	24,773	57	192
05.03	0.864	24,326	no	0.567	24,325	57	6,673	0.695	24,340	no	0.620	24,343	57	200
05.04	0.468	24,468	no	0.823	24,467	77	5,350	0.475	24,483	no	0.841	24,479	87	4,435
05.07	0.812	24,436	no	0.862	24,437	78	531	0.828	24,449	no	0.878	24,447	88	231
05.08	0.939	24,446	no	0.602	24,447	60	5,456	0.956	24,453	no	0.941	24,456	83	151
05.09	0.659	24,298	no	0.898	24,294	79	5,090	0.979	24,312	yes	0.924	24,313	83	0
05.10	0.812	24,689	no	0.807	24,689	75	1,740	0.886	24,704	yes	0.882	24,708	83	0
05.11	0.835	24,543	yes	0.798	24,541	76	0	0.917	24,563	yes	0.882	24,562	83	0
05.14	0.892	24,249	yes	0.811	24,250	78	0	0.969	24,271	yes	0.890	24,273	83	0
05.15	0.828	24,294	yes	0.775	24,297	76	0	0.896	24,313	yes	0.869	24,314	83	0
05.16	0.860	24,398	no	0.768	24,397	74	867	0.932	24,420	yes	0.838	24,423	83	0
05.17	0.684	24,620	no	0.766	24,616	76	1,473	0.823	24,637	no	0.828	24,638	83	111
05.18	0.931	24,666	yes	0.593	24,665	60	0	0.979	24,687	yes	0.920	24,689	83	0
05.21	0.855	24,830	no	0.565	24,830	61	5,231	0.900	24,853	no	0.893	24,853	83	168
05.22	0.545	24,960	no	0.771	24,962	87	6,394	0.870	24,988	no	0.896	24,990	84	368
05.24	0.804	24,947	no	0.854	24,946	86	1,545	0.925	24,966	no	0.930	24,967	87	292
05.25	0.885	24,752	no	0.868	24,751	87	777	0.951	24,775	no	0.937	24,776	92	391
05.28	0.889	24,686	no	0.865	24,688	86	593	0.949	24,714	no	0.926	24,715	90	159
05.29	0.901	24,373	no	0.943	24,374	83	3,300	0.989	24,402	no	0.950	24,403	90	270
05.30	0.968	24,015	no	0.919	24,013	79	2,075	0.979	24,045	no	0.963	24,044	90	296
05.31	0.769	24,292	no	0.760	24,292	75	2,112	0.814	24,325	no	0.807	24,324	89	118
06.01	0.870	24,303	no	0.817	24,303	78	2,785	0.918	24,334	no	0.893	24,337	88	97
06.04	0.719	24,462	no	0.769	24,461	75	1,009	0.811	24,497	no	0.839	24,500	88	153
06.05	0.800	24,727	no	0.811	24,726	80	773	0.863	24,765	no	0.889	24,767	88	114
06.06	0.873	24,483	no	0.487	24,484	62	6,034	0.968	24,534	no	0.891	24,535	87	360
06.07	0.503	24,470	no	0.817	24,469	79	5,332	0.869	24,534	yes	0.908	24,534	87	0
06.08	0.765	24,424	no	0.503	24,428	65	5,132	0.798	24,483	no	0.800	24,484	86	91
06.11	0.546	24,381	no	0.514	24,384	62	577	0.831	24,424	no	0.870	24,424	86	166
06.12	0.579	23,945	no	0.501	23,947	62	583	0.948	23,992	yes	0.892	23,993	86	0
06.13	0.381	24,166	no	0.400	24,166	59	712	0.810	24,214	no	0.796	24,216	86	42
06.14	0.556	23,865	no	0.443	23,866	59	860	0.956	23,911	no	0.829	23,912	85	355
06.15	0.390	23,584	no	0.332	23,584	59	929	0.759	23,637	no	0.748	23,637	85	84
06.18	0.278	23,298	no	0.269	23,297	54	868	0.713	23,357	yes	0.713	23,358	85	0
06.19	0.958	23,515	no	0.595	23,516	65	2,728	1.727	23,566	no	0.861	23,562	76	1,571
06.20	0.651	23,272	no	0.723	23,273	67	1,275	0.932	23,349	no	0.997	23,347	76	288
06.21	0.700	23,289	no	0.823	23,287	77	1,273	0.910	23,357	no	1.016	23,355	76	394
06.22	1.071	23,319	no	1.038	23,316	74	1,142	1.266	23,386	no	1.237	23,385	76	94
06.25	1.177	23,308	no	0.989	23,308	71	1,170	1.383	23,376	no	1.174	23,377	76	379
06.26	0.878	23,050	no	0.992	23,049	65	3,586	1.059	23,129	no	1.238	23,130	75	1,181
06.27	0.952	23,092	no	0.833	23,091	68	3,617	1.180	23,178	no	1.185	23,176	75	120
06.28	0.680	23,563	no	0.813	23,561	68	1,302	0.978	23,633	no	1.077	23,632	75	512
06.29	0.997	23,869	no	1.047	23,867	70	1,311	1.244	23,927	no	1.316	23,930	75	339

Table 4
(Continued.)

Day	$\lambda = 0$							$\lambda = 0.25$						
	Current portfolio			"Optimized" portfolio				Current portfolio			"Optimized" portfolio			
	TEV	NAV	fea	*TEV*	NAV	#	C^{tac}	*TEV*	NAV	fea	*TEV*	NAV	#	C^{tac}
07.02	0.996	24,159	no	1.110	24,159	77	1,986	1.279	24,212	no	1.393	24,212	76	343
07.03	1.317	23,985	no	1.211	23,985	72	1,157	1.595	24,036	no	1.602	24,037	76	436
07.04	1.121	23,829	no	1.094	23,827	70	791	1.534	23,891	yes	1.512	23,892	76	0
07.05	1.090	23,682	no	1.095	23,683	73	891	1.519	23,760	no	1.521	23,758	76	136
07.06	1.227	23,179	no	1.096	23,178	77	1,005	1.654	23,257	no	1.551	23,256	76	153
07.09	0.967	23,163	no	0.970	23,163	69	826	1.399	23,250	no	1.405	23,248	76	48
07.10	0.958	23,024	no	0.979	23,023	75	1,041	1.377	23,115	yes	1.404	23,115	76	0
07.11	1.087	22,893	no	1.041	22,893	72	1,023	1.474	22,978	no	1.466	22,977	76	82
07.12	0.920	23,196	no	0.988	23,197	73	1,435	1.311	23,267	no	1.360	23,265	76	356
07.13	1.120	23,281	no	1.126	23,280	71	1,573	1.498	23,346	no	1.492	23,346	76	187
07.16	1.313	23,098	no	1.165	23,097	70	1,371	1.664	23,165	no	1.506	23,164	76	274
07.17	1.091	23,025	no	1.026	23,025	79	3,717	1.413	23,093	no	1.369	23,094	76	104
07.18	1.019	22,577	no	0.976	22,578	70	3,063	1.369	22,658	no	1.357	22,659	76	49
Min	0.278	22,577		0.269	22,578	54	0	0.475	22,658		0.620	22,659	57	0
Max	1.317	24,960		1.211	24,999	87	6,673	1.727	24,988		1.602	25,004	92	4,435
Avg	0.845	23,973		0.812	23,991	71	2,156	1.063	24,020		1.036	24,038	80	289
Sum							118,554							15,888

= cardinality; fea = feasible; NAV in 1,000 Euros; C^{tac}: transaction costs.
Note that if the portfolio of day $T - 1$ is feasible for day T and is not re-optimized then the actual *TEV* of this portfolio for day T is shown.

In a first experiment we have neglected transaction costs and we have performed the following procedure: The heuristic is applied every day in the manner which we have described in the last section. In the case that the portfolio chosen on day $T - 1$ does not violate the investment guidelines on day T there is no need to restructure this so-called *current portfolio*.

Now, we assume that the fund manager restructures the current portfolio according to the proposal, i.e. the *"optimized" portfolio*, if and only if the current portfolio is infeasible. Otherwise, if the portfolio stays feasible, the fund manager does not restructure the portfolio and keeps the portfolio unchanged for another day even if the proposal results in a portfolio with smaller tracking error variance in order to save transaction costs. In a second experiment we have incorporated transaction costs into the optimization and applied the same managing strategy as in the first experiment. Note, that for May 1st, no initial portfolio is available and thus we adopt the portfolio proposed by the meta-heuristic to initialize the process.

Optimization neglecting transaction costs. First of all, it should be mentioned that the DAX30-benchmark is infeasible for all 56 days, violating each of the three bundle constraints IG_6, IG_7 and IG_8 significantly.

We have analyzed the quality of the current portfolio (i.e. the portfolio realized on day $T - 1$) with respect to the benchmark and prices on day T. It turned out that the current portfolio of day $T - 1$ becomes infeasible for day T quite regularly. After all, the current portfolio remains feasible from $T - 1$ to T only 4 times within the 56 days and this is mainly caused by violations of investment guideline IG_6.

Analyzing the quality of the "optimized" portfolios generated by our algorithm it turns out that we generate portfolios with shares that are very close to the bounds of the three bundle constraints IG_6–IG_8. And thus, already small changes in prices from day $T - 1$ to day T will result in shares which violate some of the constraints. Thus to obtain more stable portfolios it might be a good idea to strengthen the bounds for these guidelines for the optimization.

Starting from an initial budget of 25,000,000 Euro on May 1st, the net asset value of the portfolio (neglecting any deduction of transaction costs) is only 22,577,600 Euro at the end of the 2.5 months period. Yet, in the present case of a passively managed fund this reduction is not a sign of poor management. This development has to be analyzed relative to the development of the benchmark portfolio. It is well known that the DAX30 went downward significantly during that period of time. In figure 4 we have plotted the NAV-curves for the DAX30-benchmark and the tracking portfolio after performing a linear transformation of the data to the interval [0, 1].

This plot demonstrates in a very impressive way two important results which can be drawn from this case study:

- The TEP-model, i.e. minimization of the tracking error variance is an appropriate model to "track" a benchmark in passive stock fund management. Note, that the correlation between the two timeseries is about 0.995 which indicates the tracking quality.
- Our meta-heuristic provides fund management with portfolios which are of excellent quality. When neglecting transaction costs, the tracking error variance is sig-

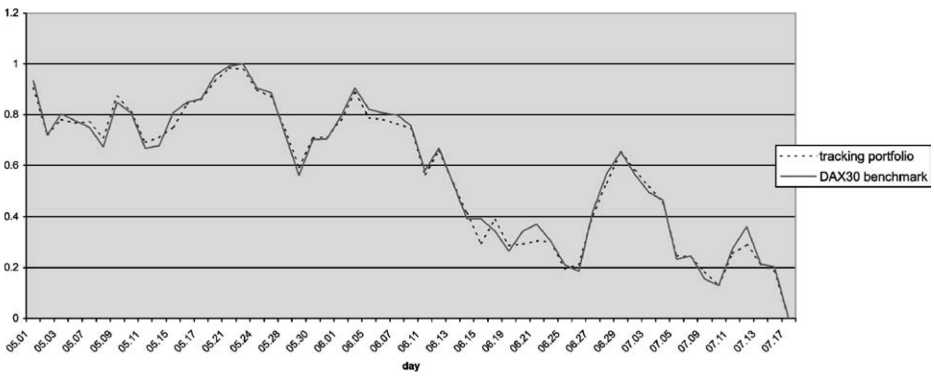


Figure 4. Quality of the tracking portfolio.

nificantly below 1, which is more than acceptable according to the fund manager's comments.

In table 4 the transaction costs are shown for the setting $\lambda = 0$ too, although the turnover volume *TO* has not been considered within this experiment. After all the costs for performing the associated buy and sell proposals amount to 2,156 Euro per day on the average and sum up to a total of approximately 118,554 Euro for the entire period. Even for a fund of 25 billion Euro this is a significant amount and this motivates to consider turnover volume and transaction costs within the optimization.

Optimization considering transaction costs. In table 4 we also report the results obtained in the second experiment when considering transaction costs in the objective function choosing $\lambda = 0.25$. Generally, the fund manager is able to modify this parameter and to perform a simulation leading to several different portfolios which are efficient with respect to tracking error variance and transaction costs. As expected the heuristic produces solutions with larger *TEV*'s than before: The increase amounts to 0.22 on average but the tracking error variances are still acceptable. This increase is compensated by a significant reduction of the turnover volume *TO* by a factor of 0.13 on average. Figure 5 shows the different developments of the turnover volume over the managing period for the two experiments.

Over the entire period total transaction costs are reduced from 118,554 Euro to 15,888 i.e. to about a seventh of the transaction costs when optimizing the portfolio with $\lambda = 0$. It is an interesting and mentionable side-effect that the number of days for which the current portfolio remains feasible rises from 4 to 12, i.e. for 21% of the 56 days a re-optimization is not favorable anymore.

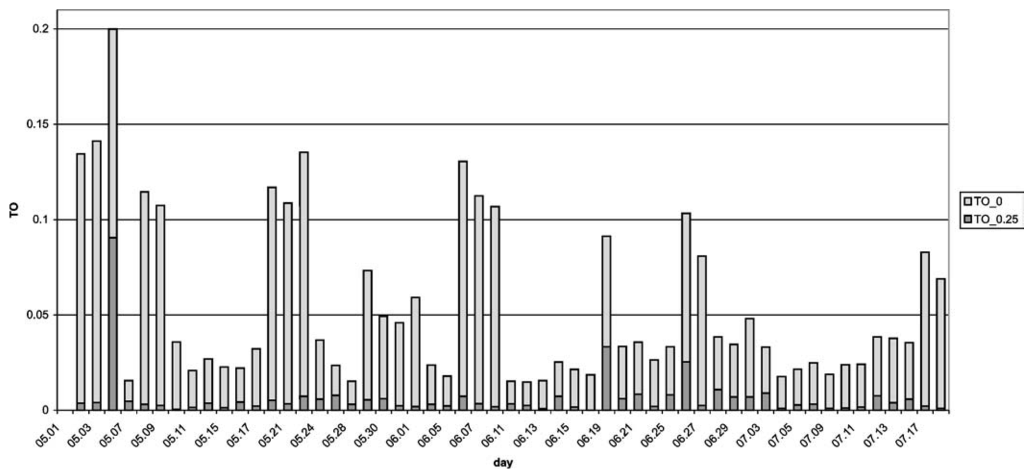


Figure 5. Comparison of the turnover volumes *TO*.

4.2. *Case II: ABC-Global*

As we have stated already, the results reported in the following are intended to allow an estimation of the absolute quality of our result. Therefore we have performed tests comparing our solution with the solution obtained by the Barrier Optimization Algorithm of CPLEX.[2]

ABC-Global is a passively managed fund with $N := N_B := 502$ assets from the index *MSCI World Developed Markets* with more than \$8 billion market capitalization each. Just as ABC-German the ABC-Global has to obey the KAGG-rules. Furthermore no short-sale of a stock and no share of more than 10% is allowed leading to a set of floor/ceiling constraints with $L_i := 0$ and $U_i := 0.1$ $(i = 1, \ldots, N)$. Also, the 502 assets in the benchmark are clustered into 51 different countries and 25 different industries and the internal guidelines for ABC-Global impose a static bundle constraint on each of these 76 groups of assets.

With the large number of assets in this benchmark the mandatory KAGG-40%-rule is relatively uncritical when prices change from day to day and thus ABC is re-optimizing the fund on a monthly basis only. In the following we will report results for a period of 43 months from February 1999 to April 2002. The monetary unit of ABC-Global is US-dollar and for reasons of confidentiality we have scaled the NAV to \$100,000,000.

In the management guidelines for ABC-Global there is no mandatory cardinality constraint imposed and thus if we neglect the dynamic bundle constraint (16) too, then we obtain a QP with linear constraints as relaxation which is solvable by standard QP-Software. Yet, there is one obstacle which does not allow to apply the exact solver of the QP-Software to this relaxation of the actual TEP-model: As for ABS-German the return model of ABC-Global is the QUANTEC factor model. An analysis of the data obtained shows that more or less constantly the matrix $D = \beta^T C \beta$ is not positive semidefinite and thus the standard exact QP-solvers are not applicable. For that reason we have separately determined the covariances for the TEV-model by a historical timeseries analysis and we have used these empirical estimators which result in positive semidefinite covariance matrices in the computational study feeding the objective function with this data by setting $K := N$, β the N-dimensional identity matrix and $V_k = Var(\epsilon_k) := 0$ for $k = 1, \ldots, N$.

Also, to create a more demanding test scenario we have artificially complicated the actual monthly problems by strengthening the bounds of the static bundle constraints systematically such that these constraints are not fulfilled by the benchmark: Let $x_l^{(B)}$ be the current weight of the benchmark for bundle l, then we draw a random number x_l' from $[0.3x_l^{(B)}, 1.7x_l^{(B)}]$ and in the case that $x_l' > x_l^{(B)}$ we set $L_l := x_l'$ and $U_l := 1$, otherwise $L_l := 0$ and $U_l := x_l'$, i.e. we make the choice of the benchmark weights infeasible for every bundle constraint.

Although the benchmark portfolio has a non-negative share for each of the 502 assets systematically applying the QP-solver to the modified problems we obtain an "optimal" cardinality which is far below $N = 502$. For our 2-phase heuristic we have imposed a cardinality constraint with $N_L := 50$ and $N_U := 150$ as in the ABC-German case.

4.2.1. Results for managing ABC-Global

Detailed results with characteristics of the solutions of the CPLEX solver and our heuristic are listed in tables 5 and 6.

After all, the 2-phase heuristic shows a behaviour which is similar to the results for the ABC-German example with a smaller number of assets: If we impose a relative small upper bound for the cardinality in phase I, $N_U = 60$ say, and the bounds $N_L = 50$ and $N_U = 150$ apply to phase II, then phase I stops with a feasible solution that has cardinality 60 at most and phase II rapidly increases this cardinality up to a value of above 300 first and then slowly reduces the cardinality to values which are close to the "optimal" cardinality proposed by the QP-solution.

In figure 6 we display the tracking error values and the cardinalities of the tracking portfolios for the 43 months/problems not considering transaction costs. These plots demonstrate that the 2-phase heuristic provides portfolios with a good quality compared to the optimal solution generated by the QP-solver. For the 43 months the relative error is only 1.65% on average with a cardinality of the heuristic solutions which is systematically larger than the cardinality of the optimal solution. This implies that when applying the heuristic the specification of a cardinality constraint seems to be advisable.

Finally figure 7 shows again the impact of transaction costs. Again we have solved the model with $\lambda = 0.25$ by our 2-phase heuristic. Detailed results are listed in table 6.

Starting from an initial value of \$100,000,000 the benchmark value has increased to about \$171,561,000 within the 3.5 years of the planning horizon. Over the entire period of 43 months the total transaction costs add up to about \$1,604,603 and \$38,204 on average for the optimal solution for the model with $\lambda = 0$ whereas solving the model with $\lambda = 0.25$ the portfolios generated by the 2-phase heuristic lead to a total value of \$391,070 and \$9,311 on the average.

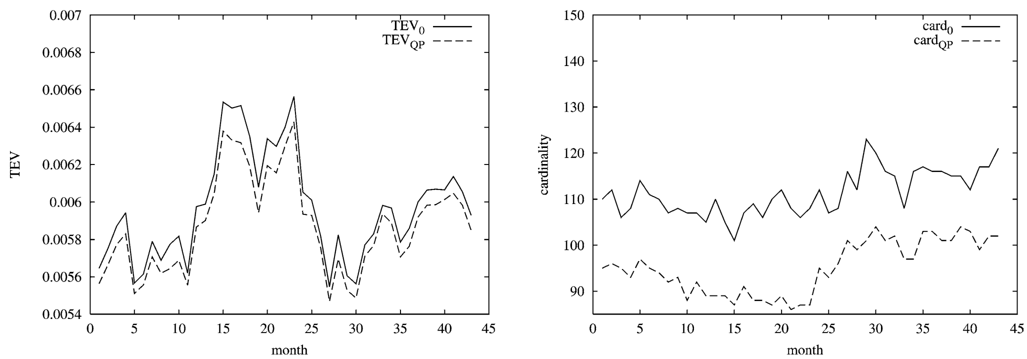


Figure 6. Comparison of *TEV* and cardinality for QP and the heuristic ($\lambda = 0$ and $\lambda = 0.25$).

Table 5

Results for ABC-Global (QP-solver and heuristic with $\lambda = 0$).

Day	BM		QP-relaxation				$\lambda = 0$							
							Current portfolio			"Optimized" portfolio				
	NAV	#	*TEV*	NAV	#	C^{tac}	*TEV*	NAV	fea	*TEV*	rE	NAV	#	C^{tac}
12.01.99	100,000	502	0.00556	100,000	95					0.00565	1.47	100,000	110	
09.02.99	106,279	502	0.00566	103,847	96	17,204	0.00559	103,627	no	0.00575	1.55	103,622	112	26,586
09.03.99	101,106	502	0.00577	98,887	95	16,633	0.00580	98,641	no	0.00587	1.72	98,638	106	25,605
06.04.99	104,206	502	0.00583	104,825	93	27,327	0.00651	104,441	no	0.00594	1.95	104,431	108	37,427
04.05.99	105,649	502	0.00551	104,204	97	17,691	0.00561	103,657	no	0.00557	1.01	103,661	114	24,190
01.06.99	109,215	502	0.00556	108,544	95	17,219	0.00572	107,863	no	0.00562	1.05	107,864	111	23,509
29.06.99	103,544	502	0.00571	101,426	94	24,800	0.00611	100,764	no	0.00579	1.45	100,764	110	27,251
27.07.99	99,295	502	0.00562	96,098	92	15,990	0.00591	95,374	no	0.00569	1.25	95,374	107	20,579
24.08.99	111,487	502	0.00564	115,758	93	35,914	0.00685	114,939	no	0.00577	2.30	114,938	108	46,338
21.09.99	119,028	502	0.00569	123,040	88	26,902	0.00592	122,524	no	0.00582	2.27	122,528	107	32,981
19.10.99	118,053	502	0.00556	121,419	92	18,183	0.00565	121,024	no	0.00562	1.15	121,026	107	22,606
16.11.99	126,054	502	0.00587	128,938	89	29,573	0.00595	128,457	no	0.00598	1.87	128,460	105	37,566
14.12.99	127,907	502	0.00590	130,910	89	21,569	0.00609	130,274	no	0.00599	1.49	130,284	110	28,470
11.01.00	122,600	502	0.00605	127,266	89	26,929	0.00657	126,555	no	0.00615	1.72	126,556	105	34,703
08.02.00	121,662	502	0.00638	127,946	87	28,728	0.00654	127,201	no	0.00653	2.44	127,193	101	34,479
07.03.00	138,413	502	0.00633	144,982	91	45,616	0.00607	143,826	no	0.00650	2.70	143,821	107	60,150
04.04.00	150,736	502	0.00632	162,318	88	84,199	0.00883	161,136	no	0.00652	3.15	161,133	109	101,476
02.05.00	147,170	502	0.00619	161,305	88	33,129	0.00693	160,111	no	0.00635	2.59	160,107	106	47,480
30.05.00	156,563	502	0.00594	166,675	87	50,711	0.00584	165,384	no	0.00608	2.30	165,381	110	65,333
27.06.00	162,363	502	0.00619	166,777	89	32,092	0.00604	165,100	no	0.00634	2.33	165,106	112	43,343
25.07.00	169,718	502	0.00616	171,337	86	49,837	0.00673	170,006	no	0.00630	2.30	170,015	108	61,431
22.08.00	185,704	502	0.00630	193,139	87	58,877	0.00644	190,770	no	0.00640	1.59	190,769	106	72,614
19.09.00	180,345	502	0.00643	194,164	87	33,582	0.00699	191,865	no	0.00656	2.13	191,860	108	48,777
17.10.00	183,415	502	0.00594	196,644	95	68,294	0.00651	194,163	no	0.00605	1.98	194,158	112	69,628
14.11.00	180,271	502	0.00593	192,532	93	39,791	0.00627	189,531	no	0.00601	1.40	189,534	107	52,681
12.12.00	179,586	502	0.00576	189,128	96	30,320	0.00587	186,343	no	0.00582	1.09	186,341	108	38,142
09.01.01	174,382	502	0.00547	188,364	101	46,006	0.00585	185,597	no	0.00555	1.43	185,584	116	56,070
06.02.01	178,330	502	0.00570	197,999	99	49,249	0.00573	195,023	no	0.00582	2.26	195,027	112	66,591

Table 5
(Continued.)

Day	BM		QP-relaxation				λ = 0							
							Current portfolio			"Optimized" portfolio				
	NAV	#	*TEV*	NAV	#	C^{tac}	*TEV*	NAV	fea	*TEV*	rE	NAV	#	C^{tac}
06.03.01	168,669	502	0.00553	191,837	101	53,554	0.00634	188,883	no	0.00561	1.40	188,884	123	65,926
03.04.01	179,994	502	0.00549	203,322	104	50,440	0.00563	200,466	no	0.00556	1.38	200,467	120	60,247
01.05.01	187,457	502	0.00571	215,608	101	57,635	0.00580	212,731	no	0.00577	1.06	212,719	116	72,164
29.05.01	184,351	502	0.00577	207,154	102	42,125	0.00596	204,084	no	0.00583	1.07	204,088	115	53,648
26.06.01	178,636	502	0.00594	204,525	97	46,432	0.00616	201,533	no	0.00598	0.75	201,526	108	50,938
24.07.01	172,973	502	0.00589	200,992	97	34,405	0.00620	198,226	no	0.00597	1.36	198,226	116	50,631
21.08.01	178,978	502	0.00571	209,544	103	41,035	0.00563	206,701	no	0.00579	1.39	206,698	117	63,263
18.09.01	170,408	502	0.00576	203,535	103	38,952	0.00611	200,601	no	0.00586	1.67	200,604	116	52,235
16.10.01	179,615	502	0.00592	214,434	101	42,002	0.00583	211,133	no	0.00600	1.37	211,124	116	64,551
13.11.01	169,346	502	0.00598	205,866	101	49,645	0.00654	202,896	no	0.00606	1.35	202,887	115	71,938
11.12.01	181,598	502	0.00599	207,649	104	36,892	0.00562	203,892	no	0.00607	1.39	203,889	115	64,606
08.01.02	180,557	502	0.00601	211,437	103	32,854	0.00650	207,701	no	0.00607	0.85	207,697	112	45,923
05.02.02	180,057	502	0.00605	202,432	99	46,984	0.00581	199,020	no	0.00614	1.46	199,014	117	57,091
05.03.02	174,382	502	0.00598	200,048	102	49,171	0.00637	196,713	no	0.00606	1.21	196,722	117	75,967
02.04.02	171,561	502	0.00585	194,262	102	36,095	0.00598	191,252	no	0.00593	1.39	191,256	121	49,702
Min	99,295	502	0.00547	96,098	86	15,990	0.00559	0		0.00555	0.75	95,374	101	20,579
Max	187,457	502	0.00643	215,608	104	84,199	0.00883	212,731		0.00656	3.15	212,719	123	101,476
Avg	151,667	502	0.00587	164,910	95	38,204	0.00618	160,698		0.00597	1.65	163,023	111	50,115
Sum						1,604,603								2,104,854

BM = Benchmark Portfolio; # = cardinality; fea = feasible; rE = relative error of heuristic *TEV* with respect to optimal *TEV* (in %); NAV in 1,000 dollars.

Table 6
Results for the ABC-Global (heuristic with λ = 0.25).

Day	$\lambda = 0.25$						
	Current portfolio			"Optimized" portfolio			
	TEV	NAV	fea	*TEV*	NAV	#	C^{tac}
12.01.99				0.00565	100,000	110	
09.02.99	0.00559	103,627	no	0.00583	103,622	115	6,833
09.03.99	0.00600	98,810	no	0.00608	98,812	118	4,843
06.04.99	0.00693	105,127	no	0.00658	105,135	122	7,240
04.05.99	0.00624	104,252	no	0.00627	104,246	122	1,960
01.06.99	0.00665	108,441	no	0.00638	108,449	121	5,480
29.06.99	0.00621	100,173	no	0.00614	100,168	122	4,768
27.07.99	0.00619	95,072	no	0.00615	95,072	123	3,459
24.08.99	0.00819	114,801	no	0.00732	114,801	125	6,188
21.09.99	0.00803	123,625	no	0.00724	123,622	123	11,410
19.10.99	0.00740	122,643	no	0.00741	122,639	122	1,927
16.11.99	0.00816	130,650	no	0.00781	130,648	118	10,350
14.12.99	0.00824	133,428	no	0.00786	133,422	115	6,810
11.01.00	0.00822	129,599	no	0.00806	129,597	117	2,527
08.02.00	0.00876	130,311	no	0.00800	130,318	122	8,576
07.03.00	0.00813	146,681	no	0.00829	146,686	117	10,479
04.04.00	0.01224	164,057	no	0.00772	164,062	121	49,895
02.05.00	0.00756	160,763	no	0.00743	160,759	124	3,659
30.05.00	0.00710	165,936	no	0.00722	165,936	118	14,986
27.06.00	0.00710	165,232	no	0.00723	165,237	119	15,814
25.07.00	0.00959	172,414	no	0.00742	172,413	124	35,068
22.08.00	0.00776	191,632	no	0.00727	191,639	120	29,285
19.09.00	0.00747	191,567	no	0.00708	191,562	126	12,037
17.10.00	0.00697	193,904	no	0.00683	193,907	123	8,961
14.11.00	0.00682	188,668	no	0.00672	188,664	122	4,356
12.12.00	0.00654	185,486	no	0.00657	185,495	121	3,795
09.01.01	0.00677	186,214	no	0.00653	186,215	121	9,908
06.02.01	0.00710	196,686	no	0.00699	196,691	123	15,541
06.03.01	0.00744	191,563	no	0.00725	191,560	123	6,679
03.04.01	0.00713	200,726	no	0.00684	200,737	123	7,954
01.05.01	0.00687	210,185	no	0.00679	210,184	126	4,839
29.05.01	0.00692	201,241	no	0.00684	201,230	129	5,262
26.06.01	0.00714	197,651	no	0.00712	197,656	129	8,015
24.07.01	0.00735	194,709	no	0.00720	194,715	130	7,983
21.08.01	0.00680	202,072	no	0.00687	202,078	121	4,722
18.09.01	0.00721	196,675	no	0.00699	196,680	122	5,042
16.10.01	0.00696	205,687	no	0.00698	205,692	122	5,221
13.11.01	0.00732	196,785	no	0.00717	196,789	134	5,038
11.12.01	0.00666	196,748	no	0.00674	196,740	130	13,044
08.01.02	0.00732	200,063	no	0.00722	200,060	129	3,401
05.02.02	0.00691	191,967	no	0.00683	191,963	133	10,101
05.03.02	0.00713	190,257	no	0.00706	190,257	130	4,101
02.04.02	0.00681	183,982	no	0.00676	183,982	129	3,494
Min	0.00559	0		0.00565	95,072	110	1,927
Max	0.01224	210,185		0.00829	210,184	134	49,895
Avg	0.00733	159,770		0.00699	162,096	123	9,311
Sum							391,070

BM = Benchmark Portfolio; # = cardinality; fea = feasible; NAV in 1,000 dollars.

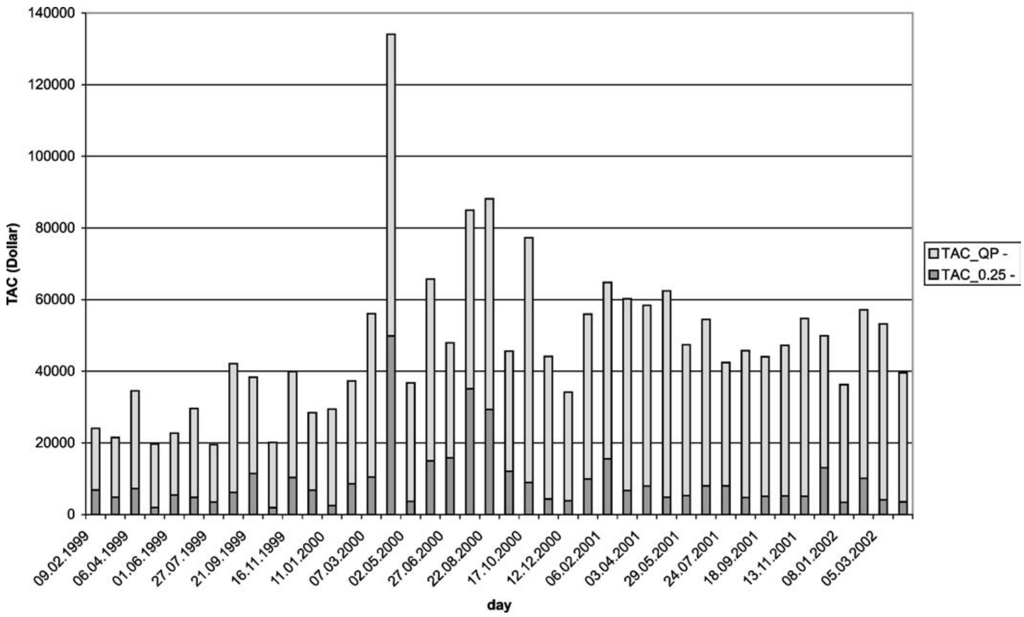


Figure 7. Comparison of the transaction costs of the heuristic solution for $\lambda = 0.25$ and the QP-solution.

5. Summary

In this paper we have introduced a 2-phase simulated annealing heuristic for solving a special class of index tracking problems in passive portfolio management. We have described the underlying model, the implementation of the two phases and, finally, we have extensively reported our experience on managing two specific funds.

The results demonstrate that in acceptable computation time the meta-heuristic approach provides solutions, i.e. proposals for the fund manager which are feasible with respect to all guidelines and which are of excellent quality with respect to the tracking error variance. We also show that through the use of the heuristic the fund manager can control the turnover volume and thus consider transaction costs in the optimization.

When applying the approach within a decision support system the fund manager may vary several (model) parameters in a very flexible and comfortable manner to analyze the stability and sensitivity of the solutions with respect to changes in the data like bounds within certain constraints, cost values, etc. This is an important feature, especially within portfolio management, since here some constraints are "soft" in the sense that they may be violated for a short period of time and recovered later. Also every portfolio manager has some individual strategic objectives and tacit knowledge which is not explicit in the "standard" optimization model but can be incorporated into the evaluation and construction via simulation.

Note that the 2-phase heuristic approach which we have presented in this paper is not immediately transferable to the five well-known portfolio optimization instances published by Beasley, Meade, and Chang (1999) on their web-page http://mscmga.ms.ic.

ac.uk/jeb/orlib/indtrackinfo.html. The reason is that the optimization model of Beasley differs significantly from our tracking error model with respect to the definition of the tracking error objective function, the return model, and the use of specialized floor/ceiling constraints.

After all, our experiences with the application of meta-heuristic based decision support tools to other portfolio management scenarios, as for instance to funds with more complex assets like futures etc., to portfolio selection models under the mean–variance paradigm with the evaluation of returns based on stochastic processes and to bond funds indicate that these approaches are practical for use in real-world problems.

Notes

1. QUANTEC *Cross-Country Model (XC)*, Release 4.0 (May–July 2001).
2. CPLEX 7.1 *Barrier Optimizer*, is a widely used professional Mathematical Programming Software package with a primal–dual predictor–corrector log barrier algorithm.

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